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LETTER TO THE EDITOR

Transitions in spectral statistics

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Abstract. We present long-range statistical properties of a recently introduced unitary random matrix ensemble, whose short-range correlations were found to describe a transition from Wigner to Poisson-type as a function of a single parameter. We argue, by evaluating the two-level correlation function of a different solvable model, that the transition is perhaps a quite general feature for a class of models. Our analytic results compare well with numerical studies on a variety of physical systems. We discuss the possibility of observing experimentally the signature of such transitions.

Statistical properties of eigenvalues of matrices describing a wide variety of quantum systems follow the universal results of random matrix models [1]. Such models were originally conceived by Wigner to provide a theoretical framework for the understanding of the statistics of energy levels of heavy nuclei [2, 3]. On very general grounds the randomness of the matrix elements, subject to any relevant symmetry requirements, gives rise to a model of eigenvalues repelling each other with a logarithmic interaction, resulting in strong correlations. The normalizability condition of the joint probability distribution requires a confining potential for the eigenvalues, which can be thought of as resulting from some physical constraint (e.g. a given eigenvalue density) [4]. For a given symmetry of the matrix, as long as the eigenvalues are well confined, the statistical properties of the levels in the bulk of the spectrum seem to be independent of the particular choice of the constraint [5], and follow a universal distribution known generally as the Wigner distribution. However, it is becoming increasingly evident that while the level statistics of a wide variety of systems can be described very well by the highly correlated Wigner distribution, many of these systems show a transition to a completely uncorrelated Poisson distribution when some relevant parameter is changed [1]. Such transitions in the spectral statistics might correspond to e.g. a chaotic–regular or a metal–insulator transition in the system. Attempts have been made to describe the transition in one particular statistical property, namely the nearest-neighbour spacing distribution (which is sensitive to only the short-range correlations between eigenvalues), either by purely heuristic interpolation schemes [6] or by considering intermediate regimes where the phase space is partly chaotic and partly regular [7]. These results differ qualitatively from the one case where the transition has been studied in detail numerically, namely the case of metal–insulator transitions in a disordered system described by the microscopic random tight-binding Anderson Hamiltonian [8]. As far as we know, no attempt has been made to explain even heuristically the transition in other statistical

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properties, such as the number variance, V_n, or the so-called Δ_3 statistics which provides a quantitative measure of the long-range rigidity of the spectrum, although there exists numerical evidence for such transitions in both chaotic [9] and disordered systems [10 11].

Recently we have argued that the appropriate random transfer matrix model related to disordered conductors belong to a new family of random matrices; a solvable model then predicted a very specific type of transition in the nearest-neighbour spacing distribution as a function of a single parameter [12]. In the present work we first show, by evaluating the two-level correlation function of a very different (though related) but still solvable model, that the nature of this transition in spectral statistics is not peculiar to the model considered in [12], but is perhaps generic for a class of models with weak confining potentials. We then calculate, for the above models, two other statistical properties which describe the transition in the long-range correlation in the bulk of the spectrum, namely the number variance and the Δ_3 statistics. We show that the variance of a linear statistic is controlled by a single parameter in these models, and is no longer universal. Our result includes the theorem of Dyson and Mehta [3] on the universality of the variance of any linear statistic as a special case in the appropriate limit. As an example, we give an explicit expression for the number variance. The nature of the transition in the Δ_3 statistics agrees with earlier numerical results for transfer matrices in disordered systems [10]. Moreover, even though the models were originally constructed for transfer matrices, numerical results for the random tight-binding Anderson Hamiltonian [11] strongly hints that the distribution of energy eigenvalues also follows a similar transition. If this is true, then the solvable models allow us to calculate and predict further consequences of such transitions in the energy level statistics. In particular, we predict how the 'correlation hole' in the Fourier transform of the absorption spectrum of small metallic particles should be destroyed with increasing disorder, and discuss the possibility of observing it experimentally.

As mentioned earlier, the random matrix models are characterized by the confining potential $V(x)$. The statistical properties of the levels can then be evaluated from the two-level correlation function which can be obtained explicitly from a set of orthogonal polynomials defined with the potential as the weight factor [3]. We will call the random matrix ensemble introduced in [12] the 'q-Hermite' unitary ensemble, because the orthogonal polynomials defined by the potential are the 'q-generalization' [13] of the classical Hermite polynomials that characterize the conventional Wigner or Gaussian unitary ensemble (GUE). The correlation function for the q-Hermite ensemble was found to depend crucially on some parameter $\beta = \ln(1/q)$ characterizing the potential

$$K(u,v) = \frac{\beta}{2\pi}\frac{\sin\pi(u-v)}{\sinh(\beta(u-v)/2)} \tag{1}$$

where the scaled variables u and v are such that the density $K(u,u)$ is unity. As noted in [12], this reduces to the corresponding correlation function for the GUE in the limit $\beta \to 0$. The above expression for the kernel is valid in the bulk of the spectrum and for $0 \leqslant \beta < 2\pi^2$. We observe here that the nearest-neighbour spacing distribution as a function of the parameter β obtained earlier for this model is remarkably similar to the one obtained numerically in [8] for the energy eigenvalues corresponding to a microscopic Anderson model for disordered conductors going through a metal–insulator transition. (The difference in the power-law behaviour for small spacing and the precise point where all different curves cross is due entirely to the fact that the numerical results are for orthogonal symmetry, while our model has unitary symmetry.) In order to establish that the above two-level kernel is not a peculiarity of the q-Hermite ensemble, we first consider another ensemble defined by the potential

$$V(x;q)=\sum_{n=0}^{\infty}\ln\left[1+(1-q)q^{n}x\right] \tag{2}$$

where the eigenvalue x is from 0 to ∞ as opposed to the range $-\infty$ to $+\infty$ for the q-Hermite ensemble. For large x and small q, this potential behaves as $[\ln x]^2$, as in the q-Hermite case, but for small x it behaves linearly as opposed to the quadratic dependence for the q-Hermite case. This potential was considered in [14] as a possible model for the transfer matrix describing disordered conductors. (Note that the density of eigenvalues at the origin for this potential do not scale with the number of eigenvalues N [14], so the usual mean-field large-N expansion breaks down). The orthogonal polynomials for this potential are the q-Laguerre polynomials, a generalization of the classical Laguerre polynomials [15]; we therefore call it the 'q-Laguerre' model. From the asymptotic properties of the q-Laguerre polynomials [14], we obtain the two-level kernel in the bulk of the spectrum as a function of $\beta=\ln(1/q)\gg 1$ in the limit where the number of eigenvalues $N\to\infty$:

$$\tilde{K}(x,y)=\frac{\mathrm{cst}}{x-y}\left[\left(\frac{x}{y}\right)^{1/4}\sin\left(\frac{\pi}{2\beta}\ln x\right)\cos\left(\frac{\pi}{2\beta}\ln y\right)-\left(\frac{y}{x}\right)^{1/4}\sin\left(\frac{\pi}{2\beta}\ln y\right)\cos\left(\frac{\pi}{2\beta}\ln x\right)\right]. \tag{3}$$

In order to compare with the Wigner distribution for which the average spacing between adjacent levels is unity, it is necessary to use a transformation of variables $x=e^{2\beta u}$, $y=e^{2\beta v}$, such that the density in the new variable is uniform and unity. In this new variable, the kernel becomes precisely the same as given in (1) for q-Hermite ensembles. It therefore follows that the statistical properties of the levels in the two models are identical in the bulk. Note that for $\beta\to 0$, the two models reduce to classical Hermite and Laguerre models, which are known to have identical kernels in the bulk [5]. (We shall not consider here the interesting 'edge effects' in the Laguerre ensemble [16].) Thus the statistical properties of the levels in the bulk of the spectrum for large N are insensitive to the details of the model as long as some general features of the model (weak $[\ln x]^2$ confinement of the large eigenvalues) remain the same.

We now evaluate the number variance and the Δ_3 statistics for the q-Hermite model, expecting that these results will be valid for at least a class of similar models. The variance $V_n=\overline{n^2}-\overline{n}^2$ of the number of eigenvalues n in an interval $(-s/2,s/2)$ can be expressed as [3]

$$V_{\mathrm{n}}(s)=\int_{-s/2}^{s/2}\mathrm{d}x\int_{-s/2}^{s/2}\mathrm{d}y\left[\delta(x-y)-Y(x-y)\right]=\frac{2}{\pi}\int_{0}^{\infty}\mathrm{d}k\,\frac{1-\cos ks}{k^2}\left[1-b(k)\right] \tag{4}$$

where the two-level form factor $b(k)$ is the Fourier transform of the two-level cluster function $Y(x)=[K(x)]^2$. For the kernel given by (1), we obtain

$$\begin{aligned}4\pi b(k)=&(|k|+2\pi)\coth[(|k|+2\pi)\pi/\beta]+(|k|-2\pi)\coth[(|k|-2\pi)\pi/\beta]\\&-2|k|\coth(|k|\pi/\beta)\,.\end{aligned} \tag{5}$$

Note that $b(k)$ reduces to the exact GUE result in the limit $\beta\to 0$. To an excellent approximation, $b(k)$ contributes negligibly to the integral in (4) for $k>2\pi$. We find

$$V_{\mathrm{n}}(s)=[1-\rho]s+\frac{\rho}{\pi^2}[\mathrm{Ci}(2\pi s)+1]+\frac{1}{\pi^2}\int_0^{2\pi s}\mathrm{d}k\,\frac{1-\cos ks}{k}\left[\coth\left(\frac{k\pi}{\beta s}\right)-\frac{\beta s}{k\pi}\right] \tag{6}$$

where Ci is the cosine integral, and ρ is defined as

$$\rho = \coth\left(\frac{2\pi^2}{\beta}\right) - \frac{\beta}{2\pi^2} \tag{7}$$

which goes to 1(0) for $\beta \to 0(\infty)$. We can obtain an explicit expression for the number variance by approximating the factor $[\coth(k\pi/\beta s) - \beta s/k\pi]$ under the integral by $k\pi/3\beta s$ up to $k = 3\beta s\rho/\pi$ and by ρ for larger k. The result

$$V_n(s) = [1-\rho]s + \frac{\rho}{\pi^2}\left[1 - \frac{\sin(3\beta\rho s/\pi)}{3\beta\rho s/\pi}\right] + \frac{\rho}{\pi^2}\left[\ln\left(\frac{2\pi^2}{3\beta\rho}\right) + \mathrm{Ci}\left(\frac{3\beta\rho s}{\pi}\right) + 1\right] \tag{8}$$

agrees with the numerical evaluation of the integral up to a small β-dependent constant for large s as well as corrections of order $1/s$. Equation (8) clearly exhibits the crucial dependence of the number variance on the parameter β. As expected, $\beta = 0$ corresponds to the exact GUE result, with a logarithmic dependence on s; increasing β corresponds to a transition towards a Poisson result, which is linear in s. Note that the variance of an arbitrary linear statistic $f = \sum_n f(x_n)$, of which V_n is a special case, will depend on the form factor (5) and is clearly no longer universal

$$\mathrm{Var}(f) = \frac{2}{\pi}\int_0^\infty \mathrm{d}k\,[1 - b(k)]|\tilde{f}(k)|^2 \tag{9}$$

where $\tilde{f}(k)$ is the Fourier transform of $f(x)$. This result generalizes the Dyson–Mehta theorem [3] on the variance of arbitrary linear statistics and reduces to it when $\beta \to 0$.

The Δ_3 statistics is a measure of the size of fluctuations of a given level sequence against a best straight line fit for that level sequence. If $s(x)$ is the staircase function for a level sequence (in the variable where the density is uniform) in a given interval, then one defines a variance $\Delta_3 = \langle \min_{A,B}[(s(u) - Au - B)^2]\rangle$, where $\langle\,\rangle$ denotes an average over an ensemble of level sequences. This can be expressed in terms of the two-level kernel $K(u, v)$ [3]. For the q-Hermite ensemble we obtain the small and large β limits explicitly

$$\Delta_3(s,\beta) = \frac{1}{2\pi}\left[\ln(2\pi s) + \gamma - \tfrac{5}{4}\right] + \frac{\beta^2 s^2}{72\pi^2} + \mathrm{O}\left((\beta s)^4\right) \tag{10}$$

for $\beta \ll 1/s$, which reduces to the GUE result for $\beta = 0$, and

$$\frac{\Delta_3(s,\beta)}{s} = \frac{1}{15}\left[\frac{\beta}{2\pi^2} - \frac{2}{\exp\left(4\pi^2/\beta\right) - 1}\right] + C + \mathrm{O}\left(\mathrm{e}^{-\beta s}\right) \tag{11}$$

for $\beta \gg 1/s$, which reduces to the Poisson result for $\beta \to \infty$. Here C is a numerical constant independent of s and very weakly dependent on β. Note that since the ratio Δ_3/s is finite, Δ_3 always has a linear dependence for large enough s, with a slope that increases with increasing β, approaching the Poisson limit for $\beta \to \infty$. Figure 1 shows the complete solution (obtained from numerical evaluation of the integrals involved) for various values of β. The deviation from the GUE result as a function of some parameter is qualitatively similar to the deviations seen numerically in both the transfer matrix [10] and energy eigenvalues [11] corresponding to the tight-binding Anderson Hamiltonian for disordered conductors, as well as for the eigenvalues of the evolution operator corresponding to the Fermi-acceleration model [9].

We find that the statistical properties of the q-Hermite (or the q-Laguerre, in the bulk) ensembles are similar to those of a wide variety of quantum systems, including energy levels of disordered conductors, studied numerically. Although we are not able to derive these ensembles from microscopic Hamiltonians at present, it is useful to explore specific spectroscopic signatures of the ensembles so that their relevance to given physical systems

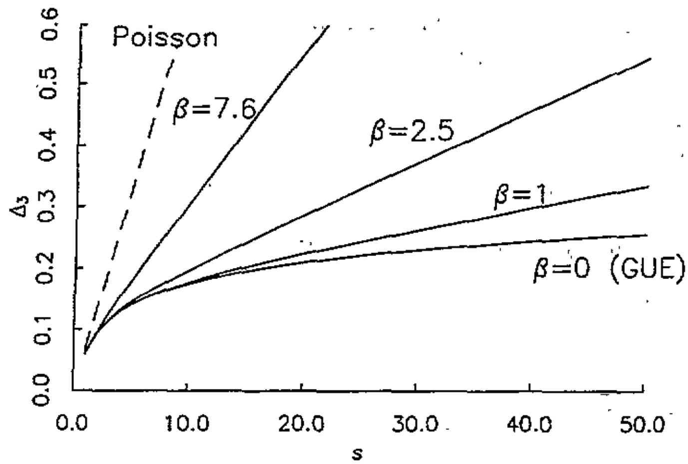

Figure 1. Δ_3 statistics as a function of the length s of a level sequence for different values of β for the q-Hermite model. The $\beta = 0$ curve coincides with the GUE result while the $\beta = \infty$ line coincides with the Poisson result.

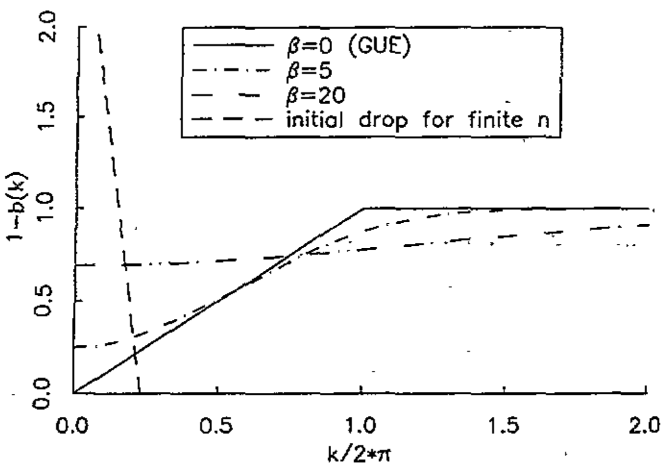

Figure 2. The function $[1 - b(k)]$ showing the change in the correlation hole with increasing β. Interpreting k as time and with the dashed line replacing an ideal vertical drop at very small time for finite number of levels n, this should correspond to the square of the Fourier transform of an experimentally obtained spectrum for a system of small disorderd particles. Increasing β will correspond to increasing disorder [12].

can be tested experimentally. The obvious problem of looking for evidence of these transitions is the difficulty to extract a 'stick spectrum' from experimental data where a large number of levels are usually lost either in the noise or in unresolved bands. One way to avoid this problem is to take direct Fourier transform (FT) of the raw experimental data; the ensemble average of the square of the FT will show a 'correlation hole' if the spectrum is chaotic, i.e. if it has a Wigner distribution [17]. The size of the correlation hole is proportional to $[1-b(k)]$, where $b(k)$ is the two-level form factor mentioned before. We plot this function for the q-Hermite ensemble obtained from (5) for various values of β in figure 2. The depth of the hole decreases from the Wigner result in a specific way towards the Poisson limit of no correlation hole ($b(k) = 0$). Thus, for example, microwave experiments on ensembles of small metallic particles (of roughly equal size) [18] at various disorders might reveal this behaviour (increasing disorder will correspond to increasing β [12]). Of course one needs sufficiently low temperature and small system size so that the energy levels are not broadened into a continuum. For a metallic system, a crude estimate within a simple electron gas model suggests that for 50 nm size particles one might expect to observe the effect at a temperature in the mK range. Because there is no contribution to the correlation hole from insulators, it might be possible to put the metallic particles in an insulating matrix.

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